MTH 516/616: TOPOLOGY II
SEMESTER 2, 2015-16

1. Homology

1.1. Simplicial Homology.

(i) Motivation for homology.
(ii) n-simplices and Δ-complexes.
(iii) The free abelian group $\Delta_n(X)$ generated by the n-simplices.
(iv) The boundary homomorphism $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$, defined by
$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|[v_0, \dots, \hat{v}_i, \dots, v_n].$$
(v) The composition $\partial_n \partial_{n-1} = 0$, and hence we have the chain complex
$$\dots \to \Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_{n-1}} \Delta_n(X) \to \dots \to \Delta_0(X) \xrightarrow{\partial_0} 0.$$
(vi) The simplicial homology group $H_n^{\Delta}(X) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$.
(vii) The simplicial homologies of S^2, $S^1 \times S^1$, $\mathbb{R}P^2$, and the Klein bottle.

1.2. Singular Homology.

(i) Singular n-simplices $\sigma : \Delta^n \to X$.
(ii) The free abelian group $C_n(X)$ of singular n-chains.
(iii) The boundary map $\partial_n(\sigma) = \sum_i \sigma|\,[v_0, \dots, \hat{v}_i, \dots, v_n]$.
(iv) The composition $\partial_n \partial_{n-1} = 0$, and hence we have the chain complex
$$\dots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \dots \to C_0(X) \xrightarrow{\partial_0} 0.$$
(v) The singular homology group $H_n(X)$.
(vi) Let $X = \sqcup_\alpha X_\alpha$, where the X_α are its path components. Then
$$H_n(X) \cong \oplus_\alpha H_n(X_\alpha).$$
(vii) If X is nonempty and path-connected , then $H_0(X) \cong \mathbb{Z}$.
(viii) If X is a point, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) \cong \mathbb{Z}$.

(ix) The augmented chain complex

$$\ldots \to C_2(X) \to C_1(X) \to C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where ϵ is defined by $\epsilon(\sum_i n_i\sigma_i) = \sum_i n_i$.

(x) The reduced homology group $\widetilde{H}_n(X)$ are the homology groups associated with the augmented chain complex.

(xi) A continuous map $f : X \to Y$ induces a homomorphism

$$f_* : H_n(X) \to H_n(Y).$$

(xii) For the composed mapping $X \xrightarrow{g} Y \xrightarrow{f} Z$, we have $(fg)_* = f_*g_*$.

(xiii) If $f, g : X \to Y$ are maps such that $f \simeq g$, then $f_* = g_*$. Consequently, $(i_X)_* = i_{H_n(X)}$.

(xiv) If $X \simeq Y$, then $H_n(X) \cong H_n(Y)$. In particular, if X is contractible, then $\widetilde{H}_n(X) = 0$ for all n.

(xv) A continuous map $f : X \to Y$ induced a homomorphism

$$f_* : H_n(X) \to H_n(Y).$$

(xvi) If $f, g : X \to Y$ are continuous maps such that $f \simeq g$, then $f_* = g_*$.

(xvii) Suppose that $f, g : X \to Y$ be continuous maps such that $f \simeq g(via\, H)$. Let $P : C_n(X) \to C_{n+1}(Y)$ be the prism operator, which is defined by

$$P(\sigma) = \sum_i F \circ (\sigma \times i_I)|[v_0 \ldots v_i, w_i, \ldots, w_n].$$

Then $\partial P + P\partial = g_\# - f_\#$.

(xviii) If $X \simeq Y$, then $H_n(X) \cong H_n(Y)$ for all n.

(xix) Properties of exact sequences.

(xx) For a pair (X, A), the group of relative n-chains

$$C_n(X, A) = C_n(X)/C_n(A).$$

(xxi) Relative homology groups $H_n(X, A)$.

(xxii) The boundary map $\partial : H_n(X, A) \to H_{n-1}(A)$.

(xxiii) The sequence of homology groups

$$\ldots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \ldots$$

is exact.

(xxiv) The sequence of reduced homology groups

$$\ldots \to \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X,A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \to \ldots$$

is exact.

(xxv) For the pair $(D^n, \partial D^n)$,

$$H_i(D^n, \partial D^n) \cong \begin{cases} \mathbb{Z}, & \text{for } i = n \\ 0, & \text{otherwise.} \end{cases}$$

(xxvi) For the pair $(X, \{x_0\})$, where $x_0 \in X$,

$$H_n(X, \{x_0\}) \cong \widetilde{H}_n(X).$$

(xxvii) Let (X, A, B) be a triple of spaces, where $B \subset A \subset X$. Then we have the following long exact sequence of homology groups:

$$\ldots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to H_{n-1}(A,B) \to \ldots$$

(xxviii) If two maps $f, g : (X, A) \to (Y, B)$ are homotopic through maps of pairs $(X, A) \to (Y, B)$. then $f_* = g_*$.

(xxix) (Excision Theorem) Given subspaces $Z \subset A \subset X$ such that $\overline{Z} \subset A^\circ$, then the inclusion $i : (X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X - Z, A - Z) \xrightarrow{i_*} H_n(X, A)$ for all n.

(xxx) Good pairs of spaces (X, A).

(xxxi) For good pairs of spaces (X, A), the quotient map $q : (X, A) \to (X/A, A/A)$ induces an isomorphism

$$q_* : H_n(X, A) \to H_n(X/A, A/A) \cong \widetilde{H}_n(X/A),$$

for all n.

(xxxii) For good pairs (X, A), $\widetilde{H}_n(X \cup CA) \cong H_n(X, A)$.

(xxxiii) For the pair $(D^n, \partial D^n)$, we have

$$H_n(D^n, \partial D^n) = \langle [i_{\Delta^n}] \rangle,$$

where i_{Δ^n} is viewed as singular a n-cycle in $C_n(D^n, \partial D^n)$.

(xxxiv) Regard S^n as a Δ-complex built from two n-simplices Δ_1^n and Δ_2^n with their boundaries identified. Then we have

$$H_n(S^n) = \langle [\Delta_1^n - \Delta_2^n] \rangle,$$

where $\Delta_1^n - \Delta_2^n$ is viewed as singular n-cycle in $C_n(S^n)$.

(xxxv) If (X, A) is a good pair of spaces, then there is an exact sequence of reduced homology groups

$$\ldots \to \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X/A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \to \ldots,$$

where $i : A \hookrightarrow X$ is the inclusion map and $j : X \to X/A$ is the quotient map.

(xxxvi) $\widetilde{H}_i(S^n) \cong \begin{cases} \mathbb{Z}, & \text{for } i = n \\ 0, & \text{otherwise.} \end{cases}$

(xxxvii) (Brouwer's fixed-point theorem) Every continuous map $f : D^n \to D^n$ has a fixed point.

(xxxviii) If a CW complex X is the union of subcomplexes A and B, then the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \to H_n(X, A)$ for all n.

(xxxix) If a wedge sum $\bigvee_\alpha$ of spaces is formed at base points $x_\alpha \in X_\alpha$ such that each pair (X_α, x_α) is good, then the inclusions $i_\alpha : H_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induces an isomorphism

$$\oplus_\alpha (i_\alpha)_* : \oplus_\alpha \widetilde{H}_n(X_\alpha) \to \widetilde{H}_n(\vee_\alpha X_\alpha).$$

(xl) If nonempty open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.

(xli) (Naturality Property) If $f : (X, A) \to (Y, b)$ is a continuous map of pairs, then the diagram

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & H_n(A) & \xrightarrow{i_*} & H_n(X) & \xrightarrow{j_*} & H_n(X,A) & \xrightarrow{\partial} & H_{n-1}(A) & \longrightarrow & \ldots \\ & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \\ \ldots & \longrightarrow & H_n(B) & \xrightarrow{i_*} & H_n(Y) & \xrightarrow{j_*} & H_n(Y,B) & \xrightarrow{\partial} & H_{n-1}(B) & \longrightarrow & \ldots \end{array}$$

is commutative.

(xlii) Let X be a Δ-complex, and A a subcomplex. Then the relative homology $H_n^\Delta(X)$ can defined using the relative chains

$$\Delta_n(X, A) = \Delta_n(X)/\Delta_n(A).$$

(xliii) For the Δ-complex pair (X, A), there exists a long exact sequence of homology groups

$$\ldots \to H_n^\Delta(A) \xrightarrow{i} H_n^\Delta(X) \xrightarrow{j} H_n^\Delta(X, A) \xrightarrow{\partial} H_{n-1}^\Delta(A) \to \ldots,$$

(xliv) Let $\phi_* : H_n^\Delta(X, A) \to H_n(X, A)$ be the canonical homomorphism induced by the chain map $\phi : \Delta_n(X, A) \to C_n(X, A)$ sending each n-simplex Δ^n of X to its characteristic map $\sigma : \Delta^n \to X$. Then ϕ_* is an isomorphism.

1.3. **Cellular homology.**

(i) The chain group $C_n^{CW}(X) = H_n(X^n, X^{n-1})$, and the chain complex

$$\dots C_{n+1}^{CW}(X) \xrightarrow{d_{n+1}} C_n^{CW}(X) \xrightarrow{d_n} C_{n-1}^{CW}(X) \to \dots,$$

where

$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$$

and

$$d_{\alpha\beta} = \deg(S_\alpha^{n-1} \to X^{n-1} \to S_\beta^{n-1})$$

that is the composition of the attaching map of e_α^n with the quotient map collapsing $X^{n-1} \setminus e_\beta^{n-1}$ to a point.

(ii) The cellular homology group is defined by

$$H_n^{CW}(X) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}.$$

(iii) If X is a CW complex, then:

(a) $H_k(X^n, X^{n-1}) = \begin{cases} 0 & \text{if } k \neq n, \text{ and} \\ \bigoplus_\alpha \langle [e_\alpha^n] \rangle & \text{if } k = n. \end{cases}$

(b) $H_k(X^n) = 0$ for $k > n$.

(c) The inclusion $i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \to H_k(X)$, if $k < n$.

(iv) $H_n^{CW}(X) \cong H_n(X)$.

1.4. **Mayer-Vietoris Sequences.**

(i) For a pair of subspaces $A, B \subset X$ such that $X = A^\circ \cup B^\circ$, there is an long exact sequence of the form

$$\dots \to H_n(A\cap B) \xrightarrow{\Phi_*} H_n(A) \oplus H_n(B) \xrightarrow{\Psi_*} H_n(X) \xrightarrow{\partial} H_{n-1}(A\cap B) \dots \to H_0(X) \to 0,$$

which is associated with the short exact sequence

$$0 \to C_n(A \cap B) \xrightarrow{\Phi} C_n(A) \oplus C_n(B) \xrightarrow{\Psi} C_n(A + b) \to 0,$$

where $\Phi(x) = (x, -x)$, and $\Psi(x, y) = x + y$.

(ii) There exists a long exact sequence identical to the one above involving reduced homology groups.

(iii) Viewing the Klein Bottle K as the union of two Mobius bands identified along their boundaries, we have that

$$H_n(K) \cong \begin{cases} \mathbb{Z}, & n = 0, \\ \mathbb{Z}, \oplus \mathbb{Z}_2 & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

1.5. **Homology with coefficients.**

(i) For a fixed abelian group G, the abelian chain groups $C_n(X; G) = \{\sum_i n_i \sigma_i \,:\, n_i \in G \text{ and } \sigma_i : \Delta^n \to X\}$.

(ii) The relative chain groups $C_n(X, A; G) = C_n(X; G)/C_n(A; G)$.

(iii) Both $C_n(X; G)$ and $C_n(X, A; G)$ form chain complexes, and the homology groups of their associated homology groups with coefficients in G are denoted by $H_n(X; G)$ and $H_n(X, A; G)$ respectively.

(iv) When $G = \mathbb{Z}_2$, n-chains are simply sums (or maybe viewed as unions) of finitely many singular n-simplices. Hence, this is the most natural tool in the absence of orientation.

(v) Mayer-Vietros sequence and the Cellular homology generalise to homology with coefficients.

(vi) If $f : S^k \to S^k$ has degree m, then $f_* : H_k(S^k; G) \to H_k(S^k; G)$ is multiplication by m.

(vii) Let F be a field of characteristic 2. Then

$$H_n(\mathbb{R}P^n; F) \cong \begin{cases} F, & 0 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

(viii) Given an abelian group G and an integer $n \geq 1$, the Moore space $M(G, n)$ is a CW- complex X satisfying

(a) $H_n(X) \cong G$ and $\widetilde{H}_i(X)$, if $i \neq n$, and

(b) X is simply-connected if $n > 1$.

(ix) The Moore space $X = M(\mathbb{Z}_m, n)$ is obtained by attached e^{n+1} to S^n by a degree m map.

1.6. **Applications of homology.**

(i) The degree of a map $f : S^n \to S^n$ denoted by $\deg f$, and its properties.

(ii) S^n has a continuous tangent vector field iff n is odd.

(iii) For n even, $\mathbb{Z}_2$ is the only nontrivial group that can act freely on S^n.

(iv) The local degree of a map $f : S^n \to S^n$ at a point x_i denoted by $\deg f|_{x_i}$.

(v) $\deg f = \sum_i \deg f|_{x_i}$.

(vi) The map $z^k : S^1 \to S^1$ has degree k.

(vii) Constructing a map $f : S^n \to S^n$ of any given degree k.

(viii) If $Sf : S^{n+1} \to S^{n+1}$ is the suspension of the map $f : S^n \to S^n$, then $\deg Sf = \deg f$.

(ix) $H_i(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, 2, \dots, 2n \\ 0 & \text{otherwise.} \end{cases}$

(x) $H_i(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0 \text{ or } i = n \text{ odd} \\ \mathbb{Z}_2 & \text{for } i \text{ odd, } 0 < i < n \\ 0 & \text{otherwise.} \end{cases}$

(xi) Let S_g denote the closed orientable surface of genus g. Then

$$H_i(S_g) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, 2 \\ \mathbb{Z}^{2g} & \text{for } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(xii) Let N_g denote the closed nonorientable surface with g crosscaps. Then

$$H_i(N_g) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 & \text{for } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(xiii) The Euler Characteristic of a finite-dimensional CW complex X having c_i i-cells for $0 \le i \le n$, is given by

$$\chi(X) = \sum_{i=0}^{n} (-1)^i c_n.$$

(xiv) If $X = X^n$ is a CW complex, then
$$\chi(X) = \sum_{i=0}^{n} (-1)^i \text{rank}\, H_n(X).$$

(xv) If $X = X^n$ and $Y = Y^n$ are CW complexes such that $X \approx Y$, then $\chi(X) = \chi(Y)$.

(xvi) If $r : X \to A$ is a retraction, then $i_* H_n(A) \to H_n(X)$ induced by the inclusion $i : A \hookrightarrow X$ is injective. Hence, we have a short exact sequence
$$0 \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \to 0$$
that splits since $r_* \circ i_* = (i_A)_*$. Consequently,
$$H_n(X) \cong H_n(A) \oplus H_n(X, A).$$

(xvii) A $K(G, 1)$ space X is a path-connected space with contractible universal cover, and which satisfies $\pi_1(X) \cong G$.

(xviii) If a finite-dimensional CW complex is a $K(G, 1)$, then the group $G = \pi_1(X)$ is torsion-free.

(xix) If D is a subspace of S^n homeomorphic to D^k for some $k \geq 0$, then $\widetilde{H}_i(S^n - D) = 0$, for all i.

(xx) (Generalised Jordan Curve Theorem). If S is a subspace of S^n homeomorphic to S^k for some k with $0 \leq k \leq n$, then
$$\widetilde{H}_i(S^n - S) \cong \begin{cases} \mathbb{Z} & \text{for } i = n - k - 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

(xxi) (Invariance of Domain) If a subspace X of $\mathbb{R}^n$ is homeomorphic to an open set in $\mathbb{R}^n$, then X is itself open in $\mathbb{R}^n$.

(xxii) If M is a compact n-manifold and N is a connected n-manifold, then an embedding $M \hookrightarrow N$ must be surjective.

(xxiii) An odd map $f : S^n \to S^n$ must have odd degree.

(xxiv) (Borsuk-Ulam Theorem) For every map $g : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ such that $g(x) = g(-x)$.

2. Singular Cohomology

(i) Motivation for cohmology.

(ii) The cochain complex C^* of free abelian groups

$$\ldots \leftarrow C^*_{n+1} \xleftarrow{\delta_{n+1}} C^*_n \xleftarrow{\delta_n} C^*_{n-1} \leftarrow \ldots,$$

is the dual of the chain complex C

$$\ldots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \ldots,$$

where for all i, $C^*_i = \operatorname{Hom}(C_i, G)$ and $\delta_i = \partial^*_i$,

(iii) The cohomology groups

$$H^n(C; G) = \operatorname{Ker} \delta_{n+1} / \operatorname{Im} \delta_n.$$

(iv) There exists a natural map $h : H^n(C^*; G) \to \operatorname{Hom}(H_n(C), G)$, which yields the following split short exact sequence

$$0 \to \operatorname{Ker} h \to H^n(C^*; G) \xrightarrow{h} \operatorname{Hom}(H_n(C), G) \to 0.$$

(v) There is a long exact sequence

$$\ldots \leftarrow B^*_n \xleftarrow{i^*_n} Z^*_n \leftarrow H^n(C^*; G) \leftarrow B^*_{n-1} \leftarrow \ldots$$

associated with the short exact sequence

$$0 \leftarrow Z^*_n \xleftarrow{j^*_n} C^*_n \xleftarrow{\delta_n} B^*_{n-1} \leftarrow 0,$$

where i^*_n and i_* are the duals of the inclusions $i_n : B_n \hookrightarrow Z_n$, and $j_n : Z_n \hookrightarrow C_n$, respectively. This long exact sequence can be expressed as the direct sum of (or can be decomposed to) the split short exact sequences

$$0 \to \operatorname{Coker} i^*_{n-1} \to H^n(C^*; G) \xrightarrow{h} \operatorname{Hom}(H_n(C), G) \to 0.$$

(vi) A free resolution F_H of an abelian group H is an exact sequence of free groups

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0.$$

The dual of the free resolution is denoted by ${F_H}^*$.

(vii) A homomomorphism $\alpha : H \to H'$ induces a chain map from $F_H \to F_{H'}$. Furthermore, any two such chain maps are chain homotopic.

(viii) For any two free resolutions F_H and F'_H of H, there are canonical isomorphisms $H^n({F_H}^*; G) \cong H^n({F'_H}^*; G)$.

(ix) Since every abelian group H has a free resolution of the form

$$0 \to F_1 \to F_0 \to H \to 0,$$

$H^n({F_H}^*; G) = 0$, for $n > 1$, and $H^n({F_H}^*; G)$ depends only on H and G, and is denoted by $\text{Ext}(H, G)$.

(x) The group $\text{Ext}(H, G)$ has the following properties.
  (i) $\text{Ext}(H \oplus H', G) \cong \text{Ext}(H, G) \oplus \text{Ext}(H', G)$.
  (ii) $\text{Ext}(H, G) = 0$, if H is free.
  (iii) $\text{Ext}(\mathbb{Z}_n, G) \cong G/nG$.

(xi) Since there is a free resolution F_H when $H = H_{n-1}(C)$

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C) \to 0,$$

its dual ${F_H}^*$

$$0 \leftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \leftarrow H_{n-1}(C)^* \leftarrow 0$$

yields the isomorphisms

$$\text{Coker}(i_{n-1}^*) \cong \text{Ext}(H_{n-1}(C), G).$$

(xii) (Universal Coefficient Theorem for Cohomology) If C is a chain complex of free abelian groups, then the cohomology groups $H^n(C; G)$ of the cochain complex C^* are determined by the split exact sequence

$$0 \to \text{Ext}(H_{n-1}(C), G) \to H^n(C^*; G) \xrightarrow{h} \text{Hom}(H_n(C), G) \to 0.$$

Consequently, we have the isomorphisms

$$H^n(C^*; G) \cong \text{Ext}(H_{n-1}(C), G) \oplus \text{Hom}(H_n(C), G).$$

(xiii) Let the homology groups $H_n = F_n \oplus T_n$ and $H_{n-1} = F_{n-1} \oplus T_{n-1}$ of a chain complex C be finitely generated abelian groups. Then

$$H^n(C^*; Z) \cong T_{n-1} \oplus (H_n/T_n).$$

(xiv) If a chain map between two chain complexes of free abelian groups induces an isomorphism of homology groups, then it induces isomorphisms of cohomology groups with any coefficient group G.

2.1. **Cup product.**

(i) Let R be a commutative ring with identity. For cochains $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, the cup product $\varphi \smile \psi$ is the cochain whose value on a singular simplex $\sigma : \Delta^{k+\ell} \to X$ is given by the formula

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|[v_0, \dots, v_k])\psi(\sigma|[v_k, \dots, v_{k+\ell}]).$$

(ii) For $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$,

$$\delta(\varphi \smile \psi)(\sigma) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi.$$

(iii) The cup product has the following properties.
  (a) The cup product of two cocycles is a cocycle.
  (b) The cup product of a cocycle and a coboundary in any order is a coboundary.
  (c) Hence the cup product induces a map at the level of cohomology

$$H^k(X;R) \times H^\ell(X;R) \xrightarrow{\smile} H^{k+\ell}(X;R),$$

  which is both associative and distributive.

(iv) For a map $f : X \to Y$, the induced maps $f^* : H^n(Y;R) \to H^n(X;R)$ satisfy

$$f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta).$$

(v) For a commutative ring R with identity, $H^*(X;R) = \oplus_k H^k(X;R)$ forms a commutative ring with identity. Furthermore, $H^*(X;R)$ is a graded ring under $\smile$.

(vi) For a graded ring A with decomposition $A = \oplus_{k\geq 0} A_k$, to indicate that $a \in A$ lies in A_k, we write $|a| = k$.

(vii) $H^*(\mathbb{R}P^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$, and $H^*(\mathbb{R}P^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]$, where $|\alpha| = 1$. In the complex case, $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$, and $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[\alpha]$, where $|\alpha| = 2$.

(viii) The inclusions $i_\alpha : X_\alpha \hookrightarrow \sqcup_\alpha X_\alpha$ induce the isomorphism

$$H^*(\sqcup_\alpha X_\alpha; R) \cong \prod_\alpha H^*(X_\alpha; R).$$

(ix) For basepoints $x_\alpha \in X_\alpha$, if (X_α, x_α) form good pairs, then we have that

$$\widetilde{H}^*(\vee_\alpha X_\alpha; R) \cong \prod_\alpha \widetilde{H}^*(X_\alpha; R).$$

(x) If R is a commutative ring, then

$$\alpha \smile \beta = (-1)^{k\ell} \beta \smile \alpha,$$

for all $\alpha \in H^k(X, A; R)$ and $\beta \in H^\ell(X, A; R)$.

(xi) $\mathbb{C}P^2$ is not homotopically equivalent to $S^2 \vee S^4$, even though they have isomorphic homology and cohomology groups.

2.2. **Orientations and homology.**

(i) The *local orientation of an n-manifold M at a point x* is a choice of generator μ_x of the group the infinite cyclic group $H_n(M, M - \{x\}) \cong \mathbb{Z}$.

(ii) Every manifold M has a two-sheeted covering space

$$\widetilde{M} = \cup_{x \in M} \{\mu_x, \mu_{-x}\}.$$

(iii) The covering space $\widetilde{M} \to M$ can be imbedded in a larger covering space $M_\mathbb{Z} \to M$ given by

$$M_\mathbb{Z} = \cup_{x \in M} \{0, \mu_{\pm x}, \mu_{\pm 2x}, \ldots\},$$

where $\mu_{kx} \leftrightarrow k \in \mathbb{Z} \cong H_n(M|x)$.

(iv) A continuous map $M \to M_\mathbb{Z}$ of the form $x \mapsto \alpha_x \in H_n(M|x)$ is called a *section* of covering space.

(v) An *orientation for M* is a section such that α_x is a generator for each x. If there exists an orientation for M, then M is said to *orientable*.

(vi) An *R-orientation for M*, where R is a commutative ring with identity, is a section of the covering space M_R that assigns to each $x \in M$, a generator $\alpha_x \in H_n(M|x; R)$.

(vii) Let M be an n-manifold. Then:

(a) $\widetilde{M}$ is orientable, and

(b) if M is connected, then M is orientable if, and only if $\widetilde{M}$ has two components. In particular, M is orientable, if its simply-connected, or more generally, if $\pi_1(M)$ has no subgroup of index 2.

(viii) An orientable manifold is R-orientable for all R, while a nonorientable manifold is R-orientable if, and only if R contains a unit of order 2. In particular, every manifold is $\mathbb{Z}_2$-orientable.

(ix) Let M be a manifold of dimension n, and let $A \subset M$ be a compact subset. Then:

(a) $H_i(M|A;R) = 0$ for $i > n$, and a class in $H_n(M|A;R)$ is zero if, and only if its image in $H_n(M|x;R)$ is zero for all $x \in A$.

(b) If $x \mapsto \alpha_x$ is a section of the covering space $M_R \to M$, then there exists a unique class $\alpha_A \in H_n(M|A;R)$ whose image in $H_n(M|x;R)$ is a α_x for all $x \in A$.

(x) Let M be a closed connected n-manifold, Then:

(a) If M is R-orientable, the map $H_n(M;R) \to H_n(M|x;R) \cong R$ is an isomorphism for all $x \in M$.

(b) If M is not R-orientable, the map $H_n(M;R) \to H_n(M|x;R) \cong R$ is injective with image $\{r \in R \,|\, 2r = 0\}$ for all $x \in M$.

(c) $H_i(M;R) = 0$, for $i > n$.

(xi) An element $[M] \in H_n(M;R)$ whose image in $H_n(M|x;R)$ is a generator for all x is called a *fundamental class.*

(xii) If M is a closed connected n-manifold, the torsion subgroup of $H_{n-1}(M;Z)$ is trivial if M is orientable and $\mathbb{Z}_2$ if M is orientable.

2.3. **Cap product and Poincaré Duality.**

(i) For an arbitrary space X and a coefficient ring R, we define an R-linear *cap product* map

$$\frown : C_k(X;R) \times C^\ell(X;R) \to C_{k-\ell}(X;R)$$

for $k \geq \ell$, by sending a singular k-simplex $\sigma : \Delta^k \to X$ and a cochain $\varphi \in C^\ell(X;R)$ to the singular $(k-\ell)$-simplex

$$\sigma \frown \varphi = \varphi(\sigma|[v_0, \dots, v_\ell])\sigma|[v_\ell, \dots, v_k].$$

(ii) The cap product has the following properties:

(a) For any $\sigma \in C_k(X;R)$ and $\varphi \in C^\ell(X;R)$,
$$\partial(\sigma \frown \varphi) = (-1)^\ell(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi).$$
(b) Cap product of a cycle and a cocycle is a cocycle.
(c) Cap product of a cycle and a coboundary is a boundary.
(d) Cap product of a boundary and a cocycle is a boundary.
(e) Thus, there is an induced cap product
$$H_k(X;R) \times H^\ell(X;R) \xrightarrow{\frown} H_{k-\ell}(X;R)$$
that is R-linear in each variable.
(f) Given a map $f : X \to Y$,
$$f_*(\alpha) \frown \varphi = f_*(\alpha \frown f^*(\varphi)).$$

(iii) (Poincaré Duality for closed manifolds) If M is a closed R-orientable n-manifold with fundamental class $[M] \in H_n(M;R)$, then the map $D : H^k(M;r) \to H_{n-k}(M;r)$ defined by $D(\alpha) = [M] \frown \alpha$ is an isomorphism for all k.

(iv) Let $C^i_c(X;G)$ be the subgroup of $C^i(X;G)$ consisting of all cochains $\varphi : C_i(X) \to G$ that are supported by a compact subset $K_\varphi \subset X$. The cohomology groups $H^i_c(X;G)$ of this subcomplex are called *cohomology groups with compact support.*

(v) Let $X_c = \{K \subset X \mid K \text{ is compact}\}$, then
$$C^i_c(X;G) = \bigcup_{K\in X_c} C^i(X, X-K;G).$$

(vi) For $K, L \in X_c$ such that $K \subset L$, the inclusion $K \hookrightarrow L$ induces inclusions $C^i(X, X-K;G) \hookrightarrow C^i(X, X-L;G)$.

(vii) Consequently, $\{H^i(X, X-K;G) \mid K \in X_c\}$ forms a directed system of groups, and we have
$$H^i_c(X;G) = \varinjlim_{K\in X_c} H^i(X, X-K;G).$$

(viii) Suppose that $X = \cup_{\alpha\in J} X_\alpha$, where J is a directed set. If for each compact $K \subset X$, there exists $\alpha = \alpha(K) \in J$ such that $K \subset X_\alpha$, then we have
$$H_i(X;G) \cong \varinjlim H_i(X_\alpha;G).$$

(ix)

$$H_c^i(\mathbb{R}^n;G) \cong \begin{cases} G, & \text{for } i = n, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

3. Homotopy Groups

(i) For a pair (X, x_0), we define

$$\pi_n(X, x_0) := \{[f] \mid f : (I^n, \partial I^n) \to (X, x_0)\}.$$

(ii) Alternatively, we can define

$$\pi_n(X, x_0) := \{[f] \mid f : (S^n, s_0) \to (X, x_0)\}.$$

(iii) When $n \geq 2$, we define an operation $+$ in $\pi_n(X, x_0)$ by:

$$(f+g)(s_1, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n), & s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, \dots, s_n), & s_1 \in [1/2, 1], \end{cases}$$

and $[f] + [g] := [f + g]$.

(iv) $(\pi_n(X, x_0), +)$ is an abelian group.

(v) Let X be a path-connected space. Given a path $\gamma : I \to X$ from x_0 to x_1, we can associate to each $f : (I^n, \partial I^n) \to (X, x_0)$ a map $f_\gamma : (I^n, \partial I^n) \to (X, x_1)$ satisfying the following properties
  (a) $(f + g)_\gamma \simeq f_\gamma + g_\gamma$.
  (b) $f_{\gamma\eta} \simeq (f_\eta)_\gamma$.
  (c) $f_e \simeq f$, where $e = e_{x_0}$.

(vi) Hence there is an induced homomorphism

$$\Phi_\gamma : (\pi_n(X, x_1), +) \to (\pi_n(X, x_0), +)$$

given by $\Phi([f]) = [f_\gamma]$ which is an isomoprhism.

(vii) A covering space $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ induces isomorphisms $p_* : \pi_n(\widetilde{X}, \widetilde{x}_0) \to \pi_n(X, x_0)$ for all $n \geq 2$. Consequently, $\pi_n(X, x_0) = 0$ for $n \geq 2$ whenever X has a contractible universal cover.

(viii) Let $(X_\alpha, x_\alpha)_{\alpha \in J}$ be an arbitrary collection path-connected spaces. Then the projection maps $p_\alpha : \prod_{\beta \in J}(X_\beta, x_\beta) \to (X_\alpha, x_\alpha)$ induces the isomorphism

$$\pi_n(\prod_{\alpha \in J}(X_\alpha, x_\alpha)) \cong \prod_{\alpha \in J} \pi_n(X_\alpha, x_\alpha).$$

(ix) For a pair of spaces (X, A) with a basepoint $x_0 \in A$ and $n \geq 1$, the *relative homotopy groups* $(\pi_n(X, A, x_0), +)$ are defined by

$$\pi_n(X, A, x_0) = \{[f] \mid f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)\},$$

where $J^{n-1} = \overline{\partial I^n - I^{n-1}}$. Alternatively, it is defined by

$$\pi_n(X, A, x_0) = \{[f] \mid f : (D^n, S^{n-1}, s_0) \to (X, A, x_0)\},$$

where the addition is done via the map $c : D^n \to D^n \vee D^n$ collapsing $D^{n-1} \subset D^n$ to a point.

(x) A map $f : (D^n, S^{n-1}, s_0) \to (X, A, x_0)$ represents zero in $\pi_n(X, A, x_0)$ if, and only if it is homotopic rel S^{n-1} to a map with image contained in A.

(xi) For a pair of spaces (X, A) with a basepoint $x_0 \in A$, the sequence

$$\ldots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \ldots$$

is exact.

(xii) For a triple of spaces (X, A, B) with $B \subset A \subset X$ and a basepoint $x_0 \in B$, the sequence

$$\ldots \to \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, B, x_0) \to \ldots$$

is exact.

(xiii) (Whitehead Theorem) Suppose that a map $f : X \to Y$ between connected CW complexes induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all n. Then:

(a) f is homotopically equivalent to Y, and

(b) furthermore if X is a subcomplex of Y, then X is a deformation retract of Y.